By considering the vertices as hinges, Thomas proves [1] that any convex quadrilateral can be deformed into a cyclic quadrilateral (having the same side lengths).

In any convex quadrilateral, Ptolemy's Inequality tells us that the product of the diagonals is less than or equal to the sum of the products of the lengths of opposite sides. In a cyclic quadrilateral, Ptolemy's Theorem, tells us that the product of the diagonals equals the sum of the products of the lengths of opposite sides. Given our four appropriate segments, a, b, c, d, there are six ways to arrange them in a convex quadrilateral. By symmetry, only three of these are distinct.

We show these three possibilities with corresponding bound on the product of the diagonals, $AC \cdot BD$:

$(AB, BC, CD, DA) = (a, b, c, d); AC \cdot BD \leq ac + bd$
$(AB, BC, CD, DA) = (a, b, d, c); AC \cdot BD \leq ad + bc$
$(AB, BC, CD, DA) = (a, c, b, d); AC \cdot BD \leq ab + cd.$
The third case gives the largest possible value (because we've placed the two largest sides opposite one another).

Algebraically,
$ac + bd \leq ab + cd \iff 0 \leq (a - d)(b - c)$
which is true by the given ordering, and
$ad + bc \leq ab + cd \iff 0 \leq (a - c)(b - d)$
which is also true by the given ordering.

Summarizing: when the four segments are arranged in a quadrilateral $ABCD$, the product of the diagonals is $\leq AB \cdot CD + BC \cdot AD$; the largest possible value for $AB \cdot CD + BC \cdot AD$ is $ab + cd$, which is achieved when a, b and c, d are opposite sides of a cyclic quadrilateral.

Reference:

1. Peter, Thomas, Maximizing the Area of a Quadrilateral, The College Mathematics Journal, Vol. 34, No. 4 (September 2003), pp. 315-316.

Also solved by Kenneth Korbin, New York, NY; David E. Manes, Oneonta, NY, and the proposers.

5495: *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania*

Let $\{x_n\}_{n\geq 1},\ x_1 = 1,\ x_n = 1 \cdot \sqrt{3!!} \cdot \sqrt[3]{5!!} \cdot \ldots \sqrt[n]{(2n-1)!!}$.
Find:

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \right).$$

Solution 1 by Moti Levy, Rehovot, Israel

$$(2n-1)!! = \frac{(2n)!}{2^n n!}. \qquad (1)$$

Using Stirling's asymptotic formula, we have

$$n! \sim \frac{n^n}{e^n}. \tag{2}$$

Applying (2) to (1) yields

$$\sqrt[n]{(2n-1)!!} \sim \left(\frac{(2n)^{2n}}{e^{2n}2^n n!}\right)^{\frac{1}{n}} = \frac{2n}{e} \tag{3}$$

Now we use (3) to approximate x_n,

$$x_n \sim \prod_{k=1}^{n} 2ke = \frac{2^n n!}{e^n} \sim \frac{2^n \frac{n^n}{e^n}}{e^n} = \frac{2^n n^n}{e^{2n}},$$

or

$$\sqrt[n]{x_n} \sim \frac{2n}{e^2}.$$

Hence,

$$\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} \sim \frac{e^2}{2}(n+1) - \frac{e^2}{2}n = \frac{e^2}{2},$$

and we conclude that $\lim\limits_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}}\right) = \frac{e^2}{2} \cong 3.694\,5.$

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that the limit of the problem equals $\frac{e^2}{2}$.

We need the following knows results for positive integers n.

$$\begin{aligned} \ln(n!) &= \left(n+\frac{1}{2}\right)\ln n - n + A + O\left(\frac{1}{n}\right), & (1)\\ \sum_{k=1}^{n}\frac{1}{k} &= \ln n + B + O\left(\frac{1}{n}\right), & (2)\\ \ln\left(1+\frac{1}{n}\right) &= \frac{1}{n} + O\left(\frac{1}{n^2}\right). & (3)\end{aligned}$$

where A and B are constants.

By (1) we have

$$\ln((2k)!) - \ln(k!) = \ln(k!) + (2\ln 2 - 1)k + \frac{\ln 2}{2} + O\left(\frac{1}{k}\right). \tag{4}$$

Next we show that

$$_n = n\ln n + (\ln 2 - 2)n + \frac{(1+\ln 2)\ln n}{2} + O(1) \tag{5}$$

In fact by (4) we have

$$\ln x_n = \sum_{k=1}^{n} \frac{\ln((2k-1)!!)}{k}. \quad = \quad \sum_{k=1}^{n} \frac{\ln((2k)!) - \ln(k!) - (\ln 2)k}{k}$$

$$= \sum_{k=1}^{n} \left(\ln k + \ln 2 - 1 + \frac{\ln 2}{2k} + O\left(\frac{1}{k^2}\right) \right)$$

$$= \ln(n!) + (\ln 2 - 1)n + \frac{\ln 2}{2} \sum_{k=1}^{n} \frac{1}{n} + O(1),$$

and (5) follows from (1) and (2).

Let $f(n) = 2\ln n - \dfrac{\ln x_n}{n}$. By (5), we obtain

$$f(n) = \ln n + 2 - 2 + O\left(\frac{\ln n}{n}\right). \qquad (6)$$

We next show that

$$f(n+1) - f(n) = \frac{1}{n} + O\left(\frac{\ln n}{n^2}\right). \qquad (7)$$

In fact

$$f(n+1) - f(n) = 2(\ln(n+1) - \ln n) - \left(\frac{\ln x_{n+1}}{n+1} - \frac{\ln x_n}{n}\right)$$

$$= 2\ln\left(1 + \frac{1}{n}\right) + \frac{\ln x_n}{n(n+1)} - \frac{\ln(2n+2)! - \ln(n+1)! - (\ln 2)(n+1)}{(n+1)^2},$$

and (7) follows readily from (3),(5) and (4). By the mean value theorem, we have

$$e^{f(n+1)} - e^{f(n)} = (f(n+1) - f(n))\, e^t, \qquad (8)$$

where t is a number lying between $f(n)$ and $f(n+1)$. By (6), both $e^{f(n+1)}$ and $e^{f(n)}$ equal $\dfrac{e^2 n}{2}\left(1 + O\left(\dfrac{\ln n}{n}\right)\right)$. Hence, by (7) and (8),

$$\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}} = e^{f(n+1)} - e^{f(n)} = \frac{e^2}{2}\left(1 + O\left(\frac{\ln n}{n}\right)\right),$$

and our claim for the limit follows.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

We will use the lemma from Solution 3 to Problem 5398 that appeared in this Column (see Nov. 2016 issue) that stated: *"If the positive sequence (p_n) is such that $\lim_{n\to\infty} \dfrac{p_{n+1}}{np_n} = p > 0$, then $\lim_{n\to\infty} \left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}\right) = \dfrac{p}{e}$."*

We let $\{p_n\}_{n\geq 1},\ p_n = \dfrac{n^{2n}}{x_n}$. Then

$$\lim_{n\to\infty} \frac{p_{n+1}}{np_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^{2n+2}}{x_n \sqrt[n+1]{(2n+1)!!}}}{n\dfrac{n^{2n}}{x_n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{2n+2}}{n^{2n+1} \sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \frac{(n+1)^{2n}(n+1)^2}{n^{2n} n \sqrt[n+1]{(2n+1)!!}}$$

$$= \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^n\right)^2 \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)^{2(n+1)}}}{\sqrt[n+1]{n^{n+1}(2n+1)!!}} = e^2 \lim_{n\to\infty} \sqrt[n]{\frac{n^{2n}}{(n-1)^n(2n-1)!!}}$$

$$\underset{root\ crition}{=} e^2 \lim_{n\to\infty} \frac{\dfrac{(n+1)^{2(n+1)}}{n^{n+1}(2n+1)!!}}{\dfrac{n^{2n}}{(n-1)^n(2n-1)!!}} = e^2 \lim_{n\to\infty} \frac{(n+1)^{2n}(n+1)^2(n-1)^n}{n^n n^{2n}(2n+1)}$$

$$= e^2 \lim_{n\to\infty} \frac{(n+1)^{2n}(n+1)^2(n-1)^n}{n^{2n} n(2n+1)n^n}$$

$$= e^2 \lim_{n\to\infty} \frac{(n+1)^{2n}}{n^{2n}} \lim_{n\to\infty} \frac{(n+1)^2}{n^{2n+1}} \lim_{n\to\infty} \frac{(n-1)^n}{n^n}$$

$$= e^2 \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^n\right)^2 \frac{1}{2} \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^2 e^2 \frac{1}{2} e^{-1}$$

$$= \frac{e^3}{2} =: p > 0, \quad \text{which implies by the lemma mentioned above,}$$

that the required limit is $\lim\limits_{\text{n}\to\infty} \left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}\right) = \dfrac{p}{e} = \dfrac{e^2}{2}$.

Editor's comment : In addition to the above solution **Bruno Salgueiro Fanego** stated that a more general form of the problem was published by the authors' of 5495 in the journal *La Gaceta de la Real Sociedad Matemática Espanãola* vol. 17 (3), 2014, pp. 523-524. (available at http://gaceta.rsme.es/abrir.phd?id=1218.) Therein they showed: If $\{a_n\}_{n\geq 1}$ is a sequence of real positive numbers such that $\lim\limits_{n\to\infty}(a_{n+1}-a_n) = a \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{\prod_{k=1}^{n+1} f(a_k)}} - \frac{(n)^2}{\sqrt[n]{\prod_{k=1}^{n} f(a_k)}}\right) = \frac{e}{ca}.$$

Letting $a_n = n$ and $f(n) = \sqrt[n]{(2n-1)!!}$ gives the desired result.

Solution 4 by Michel Bataille, Rouen, France

Let $u_n = \frac{n^2}{\sqrt[n]{x_n}} = \left(\frac{n^{2n}}{x_n}\right)^{1/n}$. We show that $\lim\limits_{n\to\infty}(u_{n+1} - u_n) = \frac{e^2}{2}$.

To this end, we first recall that $(2n-1)!! = \dfrac{(2n)!}{2^n(n!)}$ and the following asymptotic expansion as $n \to \infty$:

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + \ln(\sqrt{2\pi}) + \frac{1}{12n} + o(1/n),$$

from which we readily deduce

$$\ln\left(\frac{(2n)!}{n!}\right) = \ln((2n)!) - \ln(n!) = n\ln(n) + n(2\ln(2) - 1) + \frac{\ln(2)}{2} - \frac{1}{24n} + o(1/n).$$

Now, we have

$$\ln(u_n) = 2\ln(n) - \frac{\ln(x_n)}{n} = 2\ln(n) - \frac{1}{n}\ln\left(\prod_{k=1}^{n}\frac{1}{2}\cdot\left(\frac{(2k)!}{k!}\right)^{1/k}\right) = 2\ln(n) + \ln(2) - \frac{s_n}{n} \quad (1)$$

where $s_n = \sum\limits_{k=1}^{n}\frac{1}{k}\cdot\ln\left(\frac{(2k)!}{k!}\right)$.

Consider the sequence $\{y_n\}_{n\geq 2}$ defined by

$$y_n = s_n - n\ln(n) - (2\ln(2) - 2)n - \frac{1+\ln(2)}{2}\cdot\ln(n).$$

For $n \to \infty$, we calculate

$$\begin{aligned}
y_n - y_{n-1} &= s_n - s_{n-1} - n\ln(n) + (n-1)\ln(n-1) - (2\ln(2) - 2) + \frac{1+\ln(2)}{2}\cdot\ln\left(1 - \frac{1}{n}\right) \\
&= \frac{1}{n}\left(\ln\left(\frac{(2n)!}{n!}\right)\right) + n\ln\left(1 - \frac{1}{n}\right) - \ln(n) - \frac{1-\ln(2)}{2}\ln\left(1 - \frac{1}{n}\right) + (2 - \ln(2)) \\
&= 2\ln(2) - 1 + \frac{\ln(2)}{2n} - \frac{1}{24n^2} + o(1/n^2) + n\left(-\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} + o(1/n^3)\right) \\
&\qquad - \frac{1-\ln(2)}{2}\left(-\frac{1}{n} - \frac{1}{2n^2} + o(1/n^2)\right) + 2 - 2\ln(2) \\
&= \frac{a}{n^2} + o(1/n^2)
\end{aligned}$$

where we set $a = -\frac{1+2\ln(2)}{8}$. Thus, the series $\sum_{n=2}^{\infty}(y_n - y_{n-1})$ is convergent. Let S denotes its sum. Then, we may write $\sum_{k=2}^{n}(y_k - y_{k-1}) = S + o(1)$ and so $y_n = b + o(1)$ as $n \to \infty$ (where $b = S + y_1$).

It follows that

$$s_n = n\ln(n) + (2\ln(2) - 2)n + \frac{1+\ln(2)}{2}\cdot\ln(n) + b + o(1)$$

as $n \to \infty$.

From (1), we now obtain

$$\ln(u_n) = \ln(n) + 2 - \ln(2) - \frac{1+\ln(2)}{2}\cdot\frac{\ln(n)}{n} - \frac{b}{n} + o(1/n).$$

First, we deduce that $\ln(u_n) = \ln(n) + 2 - \ln(2) + o(1)$, hence $u_n = e^{\ln(n)+2-\ln(2)} \cdot e^{o(1)}$ and so $u_n \sim\ n \cdot \frac{e^2}{2}$. Second, the calculation of $\ln(u_{n+1}) - \ln(u_n)$ easily leads to

$$\ln(u_{n+1}) - \ln(u_n) = \ln\left(1 + \frac{1}{n}\right) + o(1/n) = \frac{1}{n} + o(1/n).$$

(Note that
$\frac{\ln(n)}{n} - \frac{\ln(n+1)}{n+1} = \frac{1}{n}\left((\ln n)\left(1 - (1+1/n)^{-1}\right) + o(1)\right) = \frac{1}{n}\left(-\frac{\ln(n)}{n} + o(1)\right) = o(1/n)$ as $n \to \infty$.)
Since

$$u_{n+1} - u_n = u_n\left(\frac{u_{n+1}}{u_n} - 1\right) = u_n\left(e^{\ln(u_{n+1})-\ln(u_n)} - 1\right)$$

we finally arrive at

$$u_{n+1} - u_n \ \sim\ u_n(\ln(u_{n+1}) - \ln(u_n)) \ \sim\ n \cdot \frac{e^2}{2} \cdot \left(\frac{1}{n}\right) \ \sim\ \frac{e^2}{2}$$

and the result follows.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Marian Ursarescu - Romania, and the proposers.

5496: *Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Let a, b, c be real numbers such that $0 < a < b < c$. Prove that:

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) \geq 2a - 2c + 3 + \sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY

For $x > 0$ we apply the known inequality $e^x > x + 1$ to $x = a - b, b - c$, and $a - c$ to get

$$e^{a-b} > a - b + 1, \quad e^{b-c} > b - c + 1, \quad e^{a-c} > a - c + 1,$$

respectively. Adding these inequalities yields

$$e^{a-b} \ + \ e^{b-c} \ + \ e^{a-c} \ > \ 2a \ - \ 2c \ + \ 3. \qquad (1)$$

For $x > y$, we see that
$e^{x-y} > (x/y)^{\sqrt{xy}} \iff x - y > \sqrt{xy}\ln(x/y) \iff \sqrt{xy} < (x-y)/(\ln x - \ln y)$, which is the left-hand member of the *logarithmic mean inequality.* Thus we have, since $0 < a < b < c$,

$$e^{b-a} > \left(\frac{b}{a}\right)^{\sqrt{ab}}, \quad e^{c-b} > \left(\frac{c}{b}\right)^{\sqrt{bc}}, \quad e^{c-a} > \left(\frac{c}{a}\right)^{\sqrt{ac}} > \left(\frac{a}{c}\right)^{\sqrt{ac}}. \qquad (2)$$